Diagonalization of compact operators in Hilbert modules over C^*-algebras of real rank zero

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Abstract

It is known that the classical Hilbert–Schmidt theorem can be generalized to the case of compact operators in Hilbert $\mathcal{A}$-modules $\mathcal{H}^*_\mathcal{A}$ over a W^*-algebra of finite type, i.e. compact operators in $\mathcal{H}^*_\mathcal{A}$ under slight restrictions can be diagonalized over $\mathcal{A}$. We show that if $\mathcal{B}$ is a weakly dense C^*-subalgebra of real rank zero in $\mathcal{A}$ with some additional property then the natural extension of a compact operator from $\mathcal{H}_\mathcal{B}$ to $\mathcal{H}^*_\mathcal{A} \supset \mathcal{H}_\mathcal{B}$ can be diagonalized with diagonal entries being from the C^*-algebra $\mathcal{B}$.

1 Introduction

Let $\mathcal{A}$ be a C^*-algebra. We consider Hilbert $\mathcal{A}$-modules over $\mathcal{A}$ [13], i.e. (right) $\mathcal{A}$-modules $\mathcal{M}$ together with an $\mathcal{A}$-valued inner product $\langle \cdot, \cdot \rangle : \mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{A}$ satisfying the following conditions:

i) $\langle x, x \rangle \geq 0$ for every $x \in \mathcal{M}$ and $\langle x, x \rangle = 0$ iff $x = 0$,

ii) $\langle x, y \rangle = \langle y, x \rangle^*$ for every $x, y \in \mathcal{M}$,

iii) $\langle \cdot, \cdot \rangle$ is $\mathcal{A}$-linear in the second argument,

iv) $\mathcal{M}$ is complete with respect to the norm $\|x\|^2 = \|\langle x, x \rangle\|_\mathcal{A}$.

By $\mathcal{M}^* = \mathrm{Hom}_\mathcal{A}(\mathcal{M}; \mathcal{A})$ we denote the $\mathcal{A}$-module dual to $\mathcal{M}$. Let $\mathcal{H}_\mathcal{A}$ be a right Hilbert $\mathcal{A}$-module of sequences $a = (a_k)$, $a_k \in \mathcal{A}$, $k \in \mathbf{N}$ such that the series $\sum a_k^* a_k$ converges in $\mathcal{A}$ in norm with the standard basis $\{e_k\}$ and let $L_n(\mathcal{A}) \subset \mathcal{H}_\mathcal{A}$ be a submodule generated by the elements $e_1, \ldots, e_n$ of the basis. An inner $\mathcal{A}$-valued product on module $\mathcal{H}_\mathcal{A}$ is given by $\langle x, y \rangle = \sum x_k^* y_k$ for $x, y \in \mathcal{A}$. A bounded operator $\mathcal{K} : \mathcal{H}_\mathcal{A} \longrightarrow \mathcal{H}_\mathcal{A}$ is called compact [7] [8], if it possesses an adjoint operator and lies in the norm closure of the linear span of operators of the form $\theta_{x,y}$, $\theta_{x,y}(z) = x\langle y, z \rangle$, $x, y, z \in \mathcal{H}_\mathcal{A}$. From now on we

suppose that the compact operator $\mathcal{K}$ is strictly positive, i.e. operator $\langle \mathcal{K}x, x\rangle$ is positive in $\mathcal{A}$ and $\mathrm{Ker}\,\mathcal{K} = 0$. It is known [14] that in the case when $\mathcal{A}$ is a W^*-algebra the inner product can be naturally prolonged to the dual module $\mathcal{H}^*_{\mathcal{A}}$.

Definition 1.1. Let $\mathcal{A}$ be a W^*-algebra. We call an operator $\mathcal{K}$ *diagonalizable* if there exist a set $\{x_i\}$ of elements in $\mathcal{H}^*_{\mathcal{A}}$ and a set of operators $\lambda \in \mathcal{A}$ such that

i) $\{x_i\}$ is orthonormal, $\langle x_i, x_j\rangle = \delta_{ij}$,

ii) $\mathcal{H}^*_{\mathcal{A}}$ coincides with the $\mathcal{A}$-module $\mathcal{M}^*$ dual to the module $\mathcal{M}$ generated by the set $\{x_i\}$,

iii) $\mathcal{K}x_i = x_i\lambda_i$,

iv) for any unitaries u_i, $u_{i+1} \in \mathcal{A}$ we have an operator inequality

$$u_i^*\lambda_i u_i \geq u_{i+1}^*\lambda_{i+1}u_{i+1}. \tag{1.1}$$

We call the elements x_i "*eigenvectors*" and the operators λ_i "*eigenvalues*" for the operator $\mathcal{K}$. It must be noticed that the "eigenvectors" and "eigenvalues" are defined not uniquely.

The problem of diagonalizing operators in Hilbert modules was initiated by R. V. Kadison in [6] and was studied in different settings in [5],[11],[4],[15] etc. In [9],[10] we have proved the following

Theorem 1.2. *If $\mathcal{A}$ is a finite σ-finite W^*-algebra then a compact strictly positive operator $\mathcal{K}$ can be diagonalized and its "eigenvalues" are defined uniquely up to unitary equivalence.*

It is well known that in the commutative case, i.e. for $\mathcal{C} = C(X)$ being a commutative C^*-algebra, compact operators cannot be diagonalized inside $\mathcal{H}_{\mathcal{C}}$ but it becomes possible if we pass to a bigger module over a bigger W^*-algebra $L^\infty(X) \supset \mathcal{C}$. It leads us to the following

Definition 1.3. Let $\mathcal{C}$ be a C^*-algebra admitting a weakly dense inclusion in a finite σ-finite W^*-algebra $\mathcal{A}$ and let $\mathcal{K}$ be a compact strictly positive operator in $\mathcal{H}_{\mathcal{C}}$. We can naturally extend $\mathcal{K}$ to the bigger module $H^*_{\mathcal{A}}$ where it will remain compact and strictly positive and by the theorem 1.2 it can be diagonalized in this module. We call a C^*-algebra $\mathcal{C}$ admitting *weak diagonalization* if the diagonal entries for any $\mathcal{K}$ in $\mathcal{H}^*_{\mathcal{A}}$ can be taken from $\mathcal{C}$ instead of $\mathcal{A}$.

Problem. Describe the class of C^*-algebras admitting weak diagonalization.

Throughout this paper we denote by $\mathcal{A}$ a finite σ-finite W^*-algebra. Denote by $\mathcal{Z} = C(Z)$ the center of $\mathcal{A}$ and by T the standard exact center-valued trace

defined on $\mathcal{A}$, $T(\mathbf{1}) = 1$. Suppose that for a C^*-subalgebra $\mathcal{B}$ of $\mathcal{A}$ the following condition holds:

($*$) for any two projections $p, q \in \mathcal{B}$ there exist in $\mathcal{B}$ equivalent (in $\mathcal{B}$) projections $r_p \sim r_q$, $r_p \leq p$, $r_q \leq q$ such that $T(r_p) = T(r_q) = \min\{T(p)(z), T(q)(z)\}$, $z \in Z$.

The purpose of this paper is to show that the class of C^*-algebras admitting weak diagonalization contains real rank zero weakly dense C^*-subalgebras of finite σ-finite W^*-algebras with the property ($*$). Recall that real rank zero ($RR(\mathcal{B}) = 0$) means [2] that every selfadjoint operator in $\mathcal{B}$ can be approximated by operators with finite spectrum, i.e. having the form $\sum \alpha_i p_i$, where $p_i \in \mathcal{B}$ are selfadjoint mutually orthogonal projections and $\alpha_i \in \mathbf{R}$. By [2] we have in this case also $RR(\mathrm{End}_{\mathcal{B}}(L_n(\mathcal{B}))) = 0$.

2 Continuity of "eigenvalues"

For the further we need to establish some continuity properties of the "eigenvalues" of compact operators in modules over W^*-algebras.

Lemma 2.1. *Let $\mathcal{K}_1 = \sum \alpha_l^{(1)} P_l^{(1)}$, $\mathcal{K}_2 = \sum \alpha_l^{(2)} P_l^{(2)}$ be strictly positive operators in $L_n(\mathcal{A})$ with finite spectrum and let $\|\mathcal{K}_1 - \mathcal{K}_2\| < \varepsilon$. Then*

i) one can find a unitary U in $L_n(\mathcal{A})$ such that it maps the "eigenvectors" of $\mathcal{K}_2$ to the "eigenvectors" of $\mathcal{K}_1$ and $\|U^*\mathcal{K}_1 U - \mathcal{K}_2\| < \varepsilon$,

ii) "eigenvalues" $\{\lambda_i^{(r)}\}$ of operators $\mathcal{K}_r$ $(r = 1, 2)$ can be chosen in such a way that $\|\lambda_i^{(1)} - \lambda_i^{(2)}\| < \varepsilon$.

Proof. As the algebra $\mathcal{A}$ can be decomposed into a direct integral of finite factors, so it is sufficient to prove the lemma for the case when $\mathcal{A}$ is a type II_1 factor (for type I_n factors lemma is trivial). Denote by $E_{\mathcal{K}}(\lambda)$ the spectral projection for the operator $\mathcal{K}$ corresponding to the set $(-\infty, \lambda)$. If τ is an exact finite trace on $\mathcal{A}$, it can be prolonged to the (infinite) trace $\bar{\tau} = \mathrm{tr} \otimes \tau$ on the algebra $\mathrm{End}_{\mathcal{A}}(\mathcal{H}_{\mathcal{A}}^*)$ and to the finite trace on a lesser algebra $\mathrm{End}_{\mathcal{A}}(L_n(\mathcal{A}))$ where we have $\bar{\tau}(\mathbf{1}) = n$. Put

$$\varepsilon_{\mathcal{K}}(\alpha) = \inf_{\bar{\tau}(E_{\mathcal{K}}(\lambda)) \geq \alpha} \lambda, \quad 0 \leq \alpha \leq n.$$

As it is shown in [12] (the continuous minimax principle) one has

$$\varepsilon_{\mathcal{K}}(\alpha) = \inf_{P \in \mathcal{P}, \, \bar{\tau}(P) \geq \alpha} \Big\{ \sup_{\xi \in \mathrm{Im}\, P, \, \|\xi\| = 1} (\mathcal{K}\xi, \xi) \Big\}, \tag{2.1}$$

where $(\cdot,\cdot)$ denotes an inner product in a Hilbert space where the algebra $\mathrm{End}_{\mathcal{A}}(L_n(\mathcal{A}))$ is represented and $\mathcal{P}$ denotes the set of projections in $\mathrm{End}_{\mathcal{A}}(L_n(\mathcal{A}))$. It follows from (2.1) that if $\|\mathcal{K}_1 - \mathcal{K}_2\| < \varepsilon$, then

$$|\varepsilon_{\mathcal{K}_1}(\alpha) - \varepsilon_{\mathcal{K}_2}(\alpha)| < \varepsilon. \tag{2.2}$$

Let $Q_i^{(r)}$ be projections on the "eigenvectors" $x_i^{(r)}$ of the operators $\mathcal{K}_r$, corresponding to the maximal "eigenvalues" $\lambda_i^{(r)}$, $\bar{\tau}(Q_i^{(r)}) = 1$. For two divisions $\{P_l^{(1)}, Q_i^{(1)}\}$ and $\{P_l^{(2)}, Q_i^{(2)}\}$ of unity given by decompositions of $\mathcal{K}_1$ and $\mathcal{K}_2$ we can construct a finer division of unity. By [16] there exist sets of mutually orthogonal projections $R_m^{(r)} \in \mathrm{End}_{\mathcal{A}}(L_n(\mathcal{A}))$ such that

i) $\bigoplus_m R_m^{(r)} = 1$,

ii) $\bar{\tau}(R_m^{(1)}) = \bar{\tau}(R_m^{(2)})$,

iii) for every m we have $R_m^{(r)} \leq Q_i^{(r)}$ or $R_m^{(r)} \leq P_j^{(r)}$ for some i or j.

Then (after renumbering) one can write the operators $\mathcal{K}_r$ in the form $\mathcal{K}_r = \sum \alpha_m^{(r)} R_m^{(r)}$ with $\alpha_1^{(r)} \leq \alpha_2^{(r)} \leq \ldots$, $\alpha_m^{(r)} \in \mathbf{R}$. It makes possible to define a unitary $U : L_n(\mathcal{A}) \longrightarrow L_n(\mathcal{A})$ such that

$$U(\operatorname{Im} R_m^{(2)}) = \operatorname{Im} R_m^{(1)}, \tag{2.3}$$

hence $U(\operatorname{Im} Q_i^{(2)}) = \operatorname{Im} Q_i^{(1)}$ so U maps the $\mathcal{A}$-modules generated by the "eigenvectors" $x_i^{(2)}$ into the modules generated by $x_i^{(1)}$, hence $Ux_i^{(2)} = x_i^{(1)} \cdot u_i = \bar{x}_i^{(1)}$ for some unitaries $u_i \in \mathcal{A}$. Put

$$n(\alpha) = \min\{n | \bar{\tau}(\bigoplus_{m \geq n} R_m^{(r)}) \geq \alpha\}.$$

Then $\varepsilon_{\mathcal{K}_r(\alpha)} = \alpha_{n(\alpha)}^{(r)}$ and it follows from (2.2) that $|\alpha_{n(\alpha)}^{(1)} - \alpha_{n(\alpha)}^{(2)}| < \varepsilon$. But changing α we obtain that

$$|\alpha_m^{(1)} - \alpha_m^{(2)}| < \varepsilon \tag{2.4}$$

for all m. Taking $\alpha = 1$ (then $i = 1$) we have

$$\mathcal{K}_r|_{\operatorname{Im} Q_1^{(r)}} = \Lambda_1^{(r)} = \sum_{m \geq n(1)} \alpha_m^{(r)} P_m^{(r)}.$$

From (2.3) and (2.4) we conclude that

$$\|U^* \Lambda_1^{(1)} U - \Lambda_1^{(2)}\| = \| \sum_{m \geq n(1)} (\alpha_m^{(1)} - \alpha_m^{(2)}) P_m^{(2)} \| \leq \varepsilon \| \bigoplus_{m \geq n(1)} P_m^{(2)} \| = \varepsilon. \tag{2.5}$$

Choosing appropriate $\lambda_1^{(r)}$ to satisfy the conditions $\Lambda_1^{(1)}\bar{x}_1^{(1)} = \bar{x}_1^{(1)}\lambda_1^{(1)}$ and $\Lambda_1^{(2)}x_1^{(2)} = x_1^{(2)}\lambda_1^{(2)}$ we obtain the estimate

$$\|\lambda_1^{(1)} - \lambda_1^{(2)}\| < \varepsilon. \tag{2.6}$$

By the same way estimates (2.5), (2.6) can be obtained for all i and it proves the lemma. $\bullet$

Corollary 2.2. *Let $\mathcal{K}_r : \mathcal{H}_{\mathcal{A}} \longrightarrow \mathcal{H}_{\mathcal{A}}$, $r = 1, 2$ be compact strictly positive operators and let $\|\mathcal{K}_1 - \mathcal{K}_2\| < \varepsilon$. Then*

i) one can find a unitary U in $\mathcal{H}^*_{\mathcal{A}}$ such that it maps the "eigenvectors" of $\mathcal{K}_2$ to the "eigenvectors" of $\mathcal{K}_1$ and $\|U^*\mathcal{K}_1U - \mathcal{K}_2\| < \varepsilon$,

ii) "eigenvalues" $\{\lambda_i^{(r)}\}$ of operators $\mathcal{K}_r$ $(r = 1, 2)$ can be chosen in such a way that $\|\lambda_i^{(1)} - \lambda_i^{(2)}\| < \varepsilon$.

Proof. Let $L_n^{(r)}(\mathcal{A}) \in \mathcal{H}^*_{\mathcal{A}}$ denotes the Hilbert submodule generated by the first n "eigenvectors" of the operator $\mathcal{K}_r$, $L_n^{(r)}(\mathcal{A}) \cong L_n(\mathcal{A})$. It was shown in [10] that the orthogonal complement to such submodule is isomorphic to $\mathcal{H}^*_{\mathcal{A}}$ and the norm of restriction of compact operator $\mathcal{K}_r$ on the orthogonal complement to $L_n^{(r)}(\mathcal{A})$ in $\mathcal{H}^*_{\mathcal{A}}$ tends to zero, henceforth it is sufficient to consider only the case of operators in $L_n(\mathcal{A})$ and there one can approximate these operators by operators with finite spectrum. $\bullet$

3 Case of $RR(\mathcal{B}) = 0$

In this section we show that C^*-algebras of real rank zero with the property $(*)$ admit weak diagonalization.

Theorem 3.1. *Let $\mathcal{B}$ be a weakly dense C^*-subalgebra in $\mathcal{A}$ with the property $(*)$ and let $RR(\mathcal{B}) = 0$. If $\mathcal{K}$ is a compact strictly positive operator in the $\mathcal{B}$-module $\mathcal{H}_{\mathcal{B}}$ then the "eigenvalues" $\{\lambda_i\}$ of diagonalization of the natural prolongation of $\mathcal{K}$ to the $\mathcal{A}$-module $\mathcal{H}^*_{\mathcal{A}}$ can be chosen in a way that $\lambda_i \in \mathcal{B}$ would hold.*

Proof is based on the results of S. Zhang [17]. By [2],[17] the operator $\mathcal{K}$ can be approximated by operators $\mathcal{K}_n \in \mathrm{End}_{\mathcal{B}}(L_n(\mathcal{B}))$ with finite spectrum. By [17], corollary 3.5 there exist such unitaries $U_n \in \mathrm{End}_{\mathcal{B}}(L_n(\mathcal{B}))$ that the operators

$$U_n^*\mathcal{K}_nU_n = \begin{pmatrix} \lambda_1^{(n)} & & 0 \\ & \ddots & \\ 0 & & \lambda_n^{(n)} \end{pmatrix}$$

are diagonal and $\lambda_i^{(n)} \in \mathcal{B}$ are operators with finite spectrum. Show that due to the property $(*)$ by an appropriate choice of such U_n one can make the condition (1.1) valid for "eigenvalues" $\{\lambda_i^{(n)}\}$. Let $\lambda_a = \sum \alpha_k q_k$, $\lambda_b = \sum \beta_l r_l$ where q_k, $r_l \in \mathcal{B}$ are projections and suppose that $a < b$ but for some m and n inequality $\beta_m > \alpha_n$ holds. Using the possibility to diagonalize projections [17] we can find projections $s_l \in \mathcal{B}$ equivalent to r_l and such that $s_l = \oplus_k s_k^{(l)}$ and $s_k^{(l)} \le q_k$. Then put

$$\lambda'_a = \sum_{k \neq n} \alpha_k q_k \oplus \sum_{l \neq m} \alpha_n s_n^{(l)} \oplus \beta_n s_n^{(m)},$$

$$\lambda'_b = \sum_{l \neq m} \beta_l s_l \oplus \sum_{k \neq n} \beta_k s_k^{(m)} \oplus \alpha_n s_n^{(m)}$$

and notice that the operators $\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$ and $\begin{pmatrix} \lambda'_1 & & 0 \\ & \ddots & \\ 0 & & \lambda'_n \end{pmatrix}$ are unitarily equivalent. After repeating this procedure for all cases when $\beta_l > \alpha_k$ we obtain validity of (1.1) for λ'_a and λ'_b. By the same way we can order all "eigenvalues" of $\mathcal{K}_n$ remaining in $\mathcal{B}$. But by the property $(*)$ if $\|\mathcal{K}_n - \mathcal{K}_{n-1}\| < \varepsilon_n$ then one can find such unitaries $u_{i,n}$ in $\mathcal{B}$ that

$$\|u_{i,n}^* \lambda_i^{(n)} u_{i,n} - \lambda_i^{(n-1)}\| < \varepsilon_n \tag{3.1}$$

. Then $u_{i,n}^* \lambda_i^{(n)} u_{i,n} \in \mathcal{B}$. Taking a subsequence of $\{\mathcal{K}_n\}$ if necessary we can take in (3.1) $\varepsilon_n = \frac{1}{2^n}$. Then the sequence

$$\bar{\lambda}_i^{(1)} = \lambda_i^{(1)},\ \bar{\lambda}_i^{(2)} = u_{i,2}^* \lambda_i^{(2)} u_{i,2},\ \bar{\lambda}_i^{(3)} = u_{i,3}^* u_{i,2}^* \lambda_i^{(3)} u_{i,2} u_{i,3},\ \ldots$$

is fundamental in $\mathcal{B}$. Denote its limit by $\bar{\lambda}_i \in \mathcal{B}$. By the corollary 2.2 for all $\mathcal{K}_n$ we can find unitaries U_n which map the first n "eigenvectors" of $\mathcal{K}$ to "eigenvectors" of $\mathcal{K}_n$. Put $\mathcal{K}'_n = U_n^* \mathcal{K}_n U_n \in \mathrm{End}_{\mathcal{A}}(\mathcal{H}_{\mathcal{A}}^*)$. Then we have

$$\mathcal{K}'_n x_i = x_i \bar{\lambda}_i^{(n)} \tag{3.2}$$

and $\|\mathcal{K}'_n - \mathcal{K}\| \to 0$. Taking limit in (3.2) we obtain $\mathcal{K} x_i = x_i \bar{\lambda}_i$, hence $\bar{\lambda}_i$ are "eigenvalues" of $\mathcal{K}$. •

Notice that the condition $(*)$ is necessary for a C^*-algebra to have the weak diagonalization property. Indeed if $\mathcal{K}$ is a direct sum of two projections, $\mathcal{K} = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}$ then the "eigenvalues" of $\mathcal{K}$ can be ordered only if the "common part" of $1 - p$ and q lies in $\mathcal{B}$.

Remark. In the case of C^*-algebras A_θ of irrational rotation one has $RR(A_\theta) = 0$ (cf [3]) and the property $(*)$ is valid, so the theorem 3.1 gives

the answer to the problem of [10] where we have considered the Schrödinger operator in magnetic field with irrational magnetic flow. It is known that this operator can be viewed as an operator acting in a Hilbert A_θ-module. As we can imbed A_θ in a type II_1 factor $\mathcal{A}$ as a weakly dense subalgebra [1] so we can diagonalize this operator in a Hilbert $\mathcal{A}$-module. The present paper shows that the "eigenvalues" of this operator can be chosen to be elements of A_θ. So this situation is a noncommutative analogue of the case $\theta = 1$ when the corresponding operator can be diagonalized over W^*-algebra $L^\infty(\mathbf{T}^2)$ but the diagonal elements lie in a lesser C^*-algebra $C(\mathbf{T}^2)$. Notice that in case of rational θ this operator is also diagonalizable.

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